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# *Financial leverage strategy with transaction costs*

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This paper offers a class of diffusion models that mimic the firm's pecking order behaviour and are designed to optimize an intertemporal leverage strategy in the presence of refinancing transaction costs. The proposed class of models is compatible with traditional static tradeoff theories and can be used to recast those theories in a dynamic framework by superimposing refinancing costs. We derive analytical expressions for the parameters of an optimal leverage strategy with exogenous refinancing limits, including the minimum cost of capital in a stochastic dynamic framework with transaction costs, the target values to which the leverage should be readjusted when the limits are reached, and the mean leverage implied by the optimal strategy. Our class of models enriches the pecking order theory and provides a quantitative framework for its implementation as a decision tool. It also provides additional hypotheses for empirical validation of that theory. Symmetrically, our results show the importance of dynamic factors in designing and interpreting empirical tests of static tradeoff theories.

**Keywords:** dynamic diffusion model, financial leverage estimation, financial leverage strategy, pecking order theory

## 1. Introduction

Under traditional capital structure theories, the firm is faced with a flat or a U-shaped cost of capital function describing the penalty associated with deviation from a possible unique optimal interior financial leverage. The precise shape and location of that function reflects a static tradeoff among various factors based in the economy or unique to the industry or the individual firm. Despite valuable contributions of those theories to the science of finance, a strong case is made by Myers (1984) that they 'don't seem to explain the actual financing behaviour, and it seems presumptuous to advise firms on optimal capital structure when we are so far from explaining actual decisions'.[1] In large measure, this shortcoming is blamed on the manner in which static theories treat financing transaction costs. While traditional theories recognize issuance and repurchase costs of debt and equity in a deterministic static framework, the management of capital structure must deal with those costs in a dynamic framework where they are triggered by the firm's uncertain demand for external funds. The static solution for a friction-free market, by which all incremental external financing is at the static-optimal leverage ratio, is obviously non-optimal where substantial transaction costs are present. This problem is recognized but not dealt with by the traditional literature.

[1] From his Presidential Address before the American Finance Association, December 1993, p.575.

The importance of transaction costs in shaping the firm's capital structure policy is the substance of the pecking order theory advanced by Myers (1984) and Myers and Majluf (1984). According to this theory, the firm's dividend policy is set to accommodate long-term growth opportunities, and is therefore sticky in the short term. Sources of cash available to offset frequent changes in operating cash flow and meet required changes in investment are selected based on their respective transaction costs, a concept broadly defined. Internal equity is used first because it is free of such costs. When exhausted, it is followed by incremental borrowing characterized by low transaction costs. Beyond a certain limit, borrowing is followed by costly refinancing through the issuance of stock. A similar sequence of changes in the opposite direction is followed in response to an incremental excess cash flow. Consistent with evidence reported below, that incremental external equity, but not debt, has a large component of fixed transaction costs, the firm following this 'pecking order', would allow its debt and debt ratio to fluctuate within wide limits. Stock would be issued or repurchased for the purpose of refinancing only infrequently, in lump sums, when those limits were reached. This scenario is largely confirmed by evidence reported by Shyam-Sunder and Myers (1992), that frequent incremental changes observed in book debt ratios of US firms closely match contemporaneous changes in the demand for external funds. The latter changes are met almost entirely by incremental borrowing or debt retirement. Myers's (1984) earlier hypothesis about the role of transaction costs in inducing asymmetric changes in the firm's debt and equity, and the implication of this behaviour for capital structure theory, are summarized by his own statement:

> Large adjustment costs could possibly explain the observed wide variation in actual debt ratios, since firms would be forced into long excursions away from their optimal debt ratios ... If adjustment costs are large, so that some firms take extended excursions away from their targets, then we ought to give less attention to refining our static tradeoff stories and relatively more to understanding what the adjustment costs are, why they are so important, and how rational managers would respond to them (p.578).

We respond to this call by offering here a class of diffusion models which mimic the firm's financial hierarchy and are designed to optimize an intertemporal leverage strategy in the presence of refinancing transaction costs. The proposed class of models is compatible with existing static trade-off theories and can be used to recast those theories in a dynamic framework by superimposing refinancing costs.

Other existing models of the firm's dynamic leverage policy are those of Fischer, Heinkel and Zechner (FHZ, 1989), Mauer and Triantis (MT, 1994), and Bagley, Ghosh and Yaari (BGY, 1993). Our earlier work with Dilip Ghosh, which analyses the important but special case of purely fixed transaction costs, is extended here to treat transaction costs in general.

Our differences with FHZ and MT concern the theoretical framework, the scenario modelled, and the results. While they use contingent claims valuation models to study the specialized static scenario of leverage indifference cum corporate income tax shield, the $(S, s)$ diffusion model used here can assimilate the cost of capital function of any static theory. Consistent with their choice of a theoretical framework, they rely on the assumption that the firm's incremental investment is financed exclusively from the shareholders' 'deep pocket': external funds in excess of cash reserves and current earnings can come only from cuts in current dividends. While they disallow incremental borrowing between refinancing transactions, we follow the preponderance of evidence and the pecking order theory in assuming that a sticky dividend policy causes incremental investment in excess of internal sources to be financed by incremental borrowing.

In further differences from FHZ and MT, our class of models allows a closed form solution and the advantages that come with it. We derive new analytical expressions for the following components of the optimal leverage strategy with exogenous refinancing limits: (a) the firm's minimum cost of capital in a stochastic dynamic framework with transaction costs; (b) the optimal target values to which the leverage should be readjusted when the limits are reached; and (c) the mean leverage implied by the optimal strategy. The analytical solution further allows us to prove a numerical result of MT, that the combination of fixed and variable refinancing costs leads to two distinct optimal leverage target ratios, one for each of the refinancing limits.

Despite differences between the theoretical frameworks and assumptions used, the general thrust of our results is similar to that of FHZ and MT. This in itself is an important finding, confirming the robustness of the results of all four studies. Jointly with the models of FHZ, MT and BGY, those presented here show the importance of dynamic factors in designing and interpreting empirical tests of static tradeoff theories. The class of models proposed by BGY and extended here makes additional contributions by enriching the pecking order theory and providing a quantitative framework for its implementation as a decision tool. It also provides additional hypotheses for empirical validation of that theory. The potential normative usefulness of our class of models is demonstrated by a numerical solution for a realistic case too complex for an analytical solution.

The remainder of this paper is organized as follows. Section 2 offers a brief survey of the evidence on transaction costs, the basis for this study; Section 3 contains the derivation of the models; Section 4 discusses implications of the models for a testable positive theory of refinancing; Section 5 provides numerical solutions for realistic scenarios defying analytical solutions; and Section 6 a conclusion.

## 2. Evidence of transaction costs

The pecking order theory of Myers (1984) and Myers and Majluf (1984) and its empirical support by Shyam-Sunder and Myers (1992) together make a strong case for the existence of substantial, broadly defined equity transaction costs. In combination with the evidence that stock issuance and repurchase – unlike incremental changes in the firm's debt – are infrequent and in lump sums, the pecking order theory also implies the presence of a significant component of fixed equity transaction costs. Comprehensive estimates of stock issuance costs are reported in studies conducted on established US industrial firms traded on organized exchanges. Studying underwritten cash offerings, Asquith and Mullins (1986), Masulis and Korwar (1986), Mikkelson and Partch (1986) and others cited by Smith (1986) report a permanent downward adjustment in the price of the stock held by existing shareholders reaching nearly a third of the new money raised. The importance of a fixed cost component is suggested by the absence of a significant cross-sectional relationship between issue size and the associated stock price adjustment. Given the strong association between firm size and issue size, dominant variable costs would cause instead a significant relationship between issue size and the negative price adjustment of existing stock. Direct evidence for the importance of fixed stock issuance costs is reported by Smith (1977) who shows substantial economies of scale in underwriters' compensation and direct administrative expenses – two important components of the overall issuance cost.

Contrary to issuance, where overall transaction costs measured by a stock price fall entail direct cost and a *related* negative information signal working in the same direction, the adverse price effect of transaction costs in repurchase may be masked by a larger, possibly *unrelated* positive price effect of a favourable signal. Empirical studies of the repurchase price premium by Masulis (1980), Dann (1981), and Vermaelen (1981) suggest that the sampled open market purchases and tender offers are largely unrelated to leverage adjustments. These studies make no attempt to estimate the effect of transaction costs on the price premium. Measuring directly one category of transaction costs, Ferris *et al.* (1978) report that US companies repurchasing shares in lump sum via tender offer pay dealers a per-share soliciting fee averaging 6%, but reaching as high as 20% of the closing market price one week prior to the offer date. If nothing else, management attention devoted to the typically large and infrequent tender offers would entail significant fixed transaction costs on top of any variable soliciting costs. Since according to evidence reported by Elton and Gruber (1968), incremental open market repurchases involve mainly variable transaction costs, the absolute and relative importance of fixed and variable costs may vary across firms depending on the repurchase method used for refinancing.

The parallel evidence on debt indicates considerably smaller issuance and, by implication, repayment costs. Eckbo (1986) reports on insignificant stock price adjustment to large debt offerings of US industrial firms. Mikkelson and Partch (1986) report that the stock price adjustment associated with debt offerings is negative but less pronounced than that associated with stock offerings. Furthermore, private placements of debt and term loans have no significant effect on stock prices. This evidence is consistent with the presence of competing low-cost methods of raising debt privately, or publicly by 'shelf registration' under the SEC's Rule 415 of 1982.

# 3. The models

Our main objective in this paper is to present a general theoretical framework in which various refinancing strategies may be explored. Our priority is to reach a closed form solution that would allow further analysis of the components of the optimal strategy – a basis for verifying numerical solutions to more complex, realistic models. The following set of assumptions represent a balance between realism and mathematical tractability.

## 3.1 *Assumptions*

(i) Two classes of securities – debt and equity – are used by the firm to raise funds. The two securities are combined into a single state-control variable, $x$.
(ii) The firm is in steady state equilibrium.
(iii) Static leverage theories enter this class of dynamic models by specifying the appropriate fixed[2] penalty function, which is a linear transformation of the standard weighted average cost of capital (WACC) of debt and equity, exclusive of deterministic transaction costs, defined over the leverage domain. The penalty function is calculated by subtracting from the WACC at any leverage the minimum WACC and multiplying by the fixed firm size.

[2] Our models can easily be generalized by introducing a random shock to the penalty function itself.

By construction, the penalty function measures the excess annual dollar capital cost over the same domain. The flat zero value penalty function used in some models below represents the static theory of leverage indifference as per Modigliani and Miller (1958) and Miller (1977). The dynamic setting created by this static scenario is faithful to the pure version of the pecking order theory of Myers (1984) and Myers and Majluf (1984). A quadratic U-shaped function used in other models below is compatible with the standard textbook WACC of the same generic form. An increasing monotonic penalty function would represent a dominant cost of financial distress, and a decreasing one a dominant corporate tax shield on interest expense.

(iv) The penalty function is assumed to incorporate any risk preference and be independent of the dynamic leverage strategies derived below. The latter assumption is consistent with results of MT, showing that the firm's operating decisions are insensitive to its debt policy. We chose to analyse the cases of no penalty or a quadratic penalty function for their mathematical convenience and the prominence of static theories espousing leverage indifference or fixed interior optimal leverage.

(v) The stochastic excess demand for external financing is met instantaneously and costlessly by adjustments in borrowing.[3] Infrequent refinancing transactions involving lump-sum offsetting changes in debt and equity are made without delay but at a cost comprising fixed and variable components. The proceeds of stock issuance are used to retire debt, and stock repurchase is financed by borrowing. Our treatment accommodates as special cases the absence of variable or fixed transaction costs at one barrier or both.

(vi) The state-control variable, $x$, that summarizes the capital structure of the firm remains theoretically unspecified. In the present context, the correct specification of this leverage index is an empirical question. To be consistent with our models, this index must be monotonically increasing with the amount of debt, and decreasing with increases in the amount of equity. This index must also follow the diffusion process described below within the relevant range of the optimal control barriers.

(vii) The leverage index state variable follows a Wiener process with constant drift, $\mu$, and diffusion parameter, $G$:

$$dx = \mu dt + \sigma d\omega \tag{1}$$

The use of this process is compatible with the theoretical arguments of Myers (1984) and the consistent evidence of studies of Stonehill *et al.* (1975), Marsh (1982), Titman and Wessels (1988), and Shyam-Sunder and Myers (1992), whereby the firm's leverage policy is dominated by the relationship between book rather than market values of debt and equity.

(viii) The process is constrained between fixed lower and upper control barriers denoted by $a$ and

[3] A direct inclusion of transaction costs of incremental debt adjustments is prevented by the continuous nature of those adjustments under the process assumed here (see assumption (vii)). Such costs can enter our class of models only indirectly. For example, if adjustments in reality are discrete, their periodic cost would generally depend on their expected frequency, size and variability, and on their fixed and variable cost parameters. Given the exogenous and constant parameters of the process, the optimal strategy of incremental debt adjustments could be determined, under certain unrestrictive assumptions, independently of the refinancing strategy derived here. Once determined, the optimal debt-adjustment policy would entail a constant expected periodic cost, having no effect on the refinancing strategy analysed here. However, that periodic cost would be added to the excess capital cost calculated here.

$b$, respectively. When the state variable hits the upper barrier, the firm makes a lump-sum adjustment in its capital structure by issuing stock and retiring bonds in the amount $b - x_b$ at a cost $C_b(b, x_b)$. This transaction returns the state variable to a fixed return point, $x_b$, between $a$ and $b$. When the state variable hits the lower barrier, the firm makes a similar adjustment by stock repurchase refinanced by borrowing in the amount $x_a - a$, at a cost $C_a(a, x_a)$. This transaction returns the state variable to the second fixed return point, $x_a$, between $a$ and $b$. In the general case, the parameters $a$, $x_a$, $x_b$, and $b$ are determined endogenously and no *a priori* assumptions are made about the relative positions of the two return points. As the state variable driven by the process moves between the two barriers, it generates instantaneous costs that are a function of the instantaneous value of the state variable along the penalty function. Note that we are allowing the optimal return points to differ from any static-optimal leverage inherent in the penalty function.

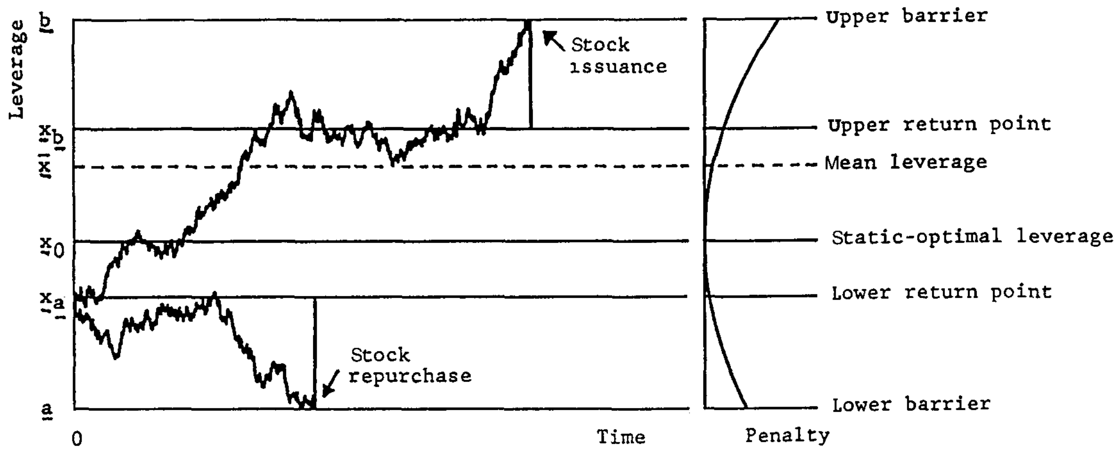


**Fig. 1.** (a) Two hypothetical time paths starting (b) Penalty function at $x_a$ and their respective equity transactions.

## 3.2 *Derivation*

Our objective is to replace the upper and lower barriers and return points for leverage readjustment so that the expected sum of periodic transaction costs and penalty for deviating from any static-optimal leverage may be minimized.

To simplify the derivation without loss of generality, we define $h = -2\mu/\sigma^2$ and substitute it into (1) to obtain the following expression for the diffusion process of our state variable, the leverage index:

$$dx = -\frac{1}{2}\sigma^2 h dt + \sigma d\omega, \quad h = \frac{-2\mu}{\sigma^2} \tag{2}$$

The derivation of the optimal refinancing strategy begins by calculating the intermediate variable $s$ and the scale function $S$ (see Karlin and Taylor, 1981 for a standard presentation of the inventory-theoretic technique)

$$s(x) = \mathrm{e}^{-s\frac{2\mu(x)}{\sigma^2(x)}\mathrm{d}x} = \mathrm{e}^{hx} \tag{3}$$

$$S(x) = \int s(x)\mathrm{d}x = \frac{\mathrm{e}^{hx}}{h} \tag{4}$$

The speed density, $m$, is calculated from (3):

$$m(x) = \frac{1}{\sigma^2 s(x)} = \frac{1}{\sigma^2 \mathrm{e}^{hx}} \tag{5}$$

From (4), we next calculate the probability of hitting the upper barrier before the lower one, $U(x)$, when starting at any point $x$ (see Fig. 1):

$$U(x) = \frac{S(x) - S(a)}{S(b) - S(a)} = \frac{\mathrm{e}^{hx} - \mathrm{e}^{ah}}{\mathrm{e}^{bh} - \mathrm{e}^{ah}}, \quad a \leqslant x \leqslant b \tag{6}$$

The expected time to reach either barrier, $V(x)$, given that the state variable starts at any $x$, is calculated based on (4)–(6):

$$\begin{aligned} V(x) &= 2\left( U(x) \int_x^b [S(b) - S(x')]m(x')\mathrm{d}x' + (1 - U(x)) \int_a^x [S(x') - S(a)]m(x')\mathrm{d}x' \right) \\ &= \frac{2((b-a)\mathrm{e}^{hx} + \mathrm{e}^{bh}(a - x) + \mathrm{e}^{ah}(x - b))}{(\mathrm{e}^{ah} - \mathrm{e}^{bh})h\sigma^2} \end{aligned} \tag{7}$$

Let $g(x)$ be a quadratic penalty function measuring instantaneous dollar costs per unit time incurred when the state variable is at any $x$:

$$g(x) = \frac{1}{2}\zeta(x - x_0)^2; \quad x_0 = 0; \quad \zeta > 0 \tag{8}$$

where the constant $\zeta/2$ measures the slope. This U-shaped function is a linear transformation of a quadratic WACC function defined over the leverage domain (see assumption (iii) above). For convenience, we set $x_0 = 0$ and $g(0) = 0$ so that the penalty function is constructed to measure only the *excess* capital cost beyond the minimum, exclusive of refinancing transaction costs. Some of the numerical results must be adjusted to reverse these shifts. When starting at any point $x$, the expected penalty incurred up to the point of encounter with either barrier is

$$\begin{aligned} W(x) &= 2\left( U(x) \int_x^b [S(b) - S(x')]m(x')g(x')\mathrm{d}x' + (1 - U(x)) \int_a^x [S(x') - S(a)]m(x')g(x')\mathrm{d}x' \right) \\ &= \frac{(6\eta h^2((a-b)\mathrm{e}^{hx} + \mathrm{e}^{ah}(b-x) + \mathrm{e}^{bh}(x-a)) + \lambda(a,b,x) + \lambda(b,x,a) + \lambda(x,a,b))}{3(-\mathrm{e}^{ah} + \mathrm{e}^{bh})h^3\sigma^2} \end{aligned} \tag{9}$$

where

$$\lambda(\gamma, \delta, \epsilon) = \mathrm{e}^{\gamma h}(6[\delta - \epsilon] + 3h[\delta^2 - \epsilon^2] + h^2[\delta^3 - \epsilon^3])\zeta \tag{10}$$

and where $\gamma$, $\delta$, and $\epsilon$ are merely place-holders. Having derived by (6) the probability of hitting the upper barrier first, $U(x)$, and that of hitting the lower barrier first, $1-U(x)$, we state the expected cost of transacting at either barrier as

$$k(x) = U(x)C_b(b-x_b) + (1-U(x))C_a(x_a-a)$$
$$= \frac{(e^{bh}-e^{hx})C_a + (e^{hx}-e^{ah})C_b}{e^{bh}-e^{ah}} \tag{11}$$

where $C_a$ and $C_b$ are any functions of the respective differences $x_a - a$ and $b - x_b$. Having calculated the expected penalty incurred up to the first encounter with either barrier, $W(x)$, and the expected transaction costs at the barriers, $k(x)$, we derive the expected periodic financing cost by dividing the combined expected cost by the expected time of reaching either barrier, $V(x)$,

$$\psi(x) = \frac{W(x)+k(x)}{V(x)}$$
$$= \frac{(\lambda(a,b,x)+\lambda(b,x,a)+\lambda(x,a,b)+3h^3\sigma^2((e^{bh}-e^{hx})C_a+(e^{hx}-e^{ah})C_b))}{6h^2(e^{bh}(x-a)+e^{ah}(b-x)+(a-b)e^{hx})} \tag{12}$$

which is valid for any starting point, $x$. In the context of the present optimization strategy, there are up to two relevant *starting* points – $x_a$ and $x_b$ – which are also the *return* points in steady-state equilibrium. To calculate the expected global costs to be incurred until the next encounter, we must determine the complementary probabilities of starting the process from point $x_a$ versus $x_b$. It is shown in Appendix A that those probabilities are, respectively, $P(a)$ and $P(b) = 1 - P(a)$, specified by

$$P(a) = \frac{d(x_b)}{1+d(x_b)-d(x_a)} = \frac{1-u(x_b)}{1+u(x_a)-u(x_b)} = \frac{e^{hx_b}-e^{bh}}{e^{ah}-e^{bh}-e^{hx_a}+e^{hx_b}} \tag{13a}$$

$$P(b) = \frac{u(x_a)}{1+u(x_a)-u(x_b)} = \frac{1-d(x_a)}{1+d(x_b)-d(x_a)} = \frac{e^{ah}-e^{hx_a}}{e^{ah}-e^{bh}-e^{hx_a}+e^{hx_b}} \tag{13b}$$

where $u(x) = 1 - d(x)$ is the probability of hitting the upper barrier first conditional upon starting the process at point $x$, and $d(x)$ the complementary conditional probability of hitting the lower barrier first. These conditional probabilities are calculated based on (6).

In its general form, the objective function to be minimized, $Z(a, x_a, x_b, b)$, is the expected periodic global cost utilizing the above expressions for the expected cost up to the next encounter and the probabilities of starting the process at points $x_a$ and $x_b$,

$$Z(a,x_a,x_b,b) = P(a)\psi(x_a) + P(b)\psi(x_b)$$
$$= \varphi(a,b) - \phi(b,a) \tag{14}$$

$$\varphi(p,q) =$$
$$\frac{(\lambda(a,b,x_p)+\lambda(b,x_p,a)+\lambda(x_p,a,b)+3h^3\sigma^2((e^{bh}-e^{hx_p})C_a+(-e^{ah}+e^{hx_p})C_b))}{6h^2(e^{ah}-e^{bh}-e^{hx_a}+e^{hx_b})((-a+b)e^{hx_p}+e^{bh}(a-x_p)+e^{ah}(-b+x_p))(e^{qh}-e^{hx_q})^{-1}} \tag{15}$$

where variables $p$ and $q$ are used as place-holders. At its minimum, $Z$ may be interpreted as the firm's *excess capital cost* inclusive of stochastic transaction costs and penalty for deviation from the static-optimal leverage. A graphical representation of the refinancing strategy is offered in Fig. 1, where two hypothetical time paths of the state variable are plotted for the sake of illustration. Since in this drawing we assume a quadratic penalty function, the apparent asymmetry of the barriers and return points with respect to the static-optimal leverage, $x_0$, would be optimal only if there is asymmetry in the process ($\mu \neq 0$) or the transaction costs ($C_a \neq C_b$).

To minimize this function under linear transaction costs, we first specify $C_a = f_a + v_a(x_a - a)$ and $C_b = f_b + v_b(b - x_b)$ and substitute these expressions in (15). We must then take the partial derivatives with respect to $a$, $x_a$, $x_b$, and $b$, set the derivatives to zero, and solve simultaneously for the optimal values of the four decision parameters. The resulting system of equations can be solved numerically but not analytically as we would hope. Being unable to obtain a solution for the general case, we are ready to settle for any case that is empirically viable.

The first scenario explored is based on the assumption of binding exogenous barriers imposed by lenders and shareholders. The objective function remains unchanged, but the set of simultaneous equations is reduced from four to two with only two unknowns, $x_a$ and $x_b$. This simplification alone does not produce a closed form solution.

In the second scenario, we reinstate the assumption of exogenous barriers and adopt the additional assumption of zero drift by taking the limit of the objective function as $h -> 0$:

$$Z_2(a, x_a, x_b, b) = \varphi(b, a) - \varphi(a, b) \tag{16}$$

where

$$\varphi(p, q) = \frac{\zeta(a^4(b - x_p) + b^4(x_p - a) - x_p^4(b - a)) + 12\sigma^2((b - x_p)C_a + (-a + x_p)C_b)(b - x_q)}{12(-a + b)(a - b - x_a + x_b)(1 - x_r)} \tag{17}$$

This simplification is justified by the argument that the drift is inconsistent with steady-state equilibrium because the firm would adjust its dividend policy to accommodate its long-term investment opportunities. Unfortunately, this simplification does not yield a closed form solution.

In the third scenario, we supplement the assumption of zero drift with the assumptions of binding exogenous barriers and no penalty function ($\zeta = 0$). This scenario, which is consistent with the leverage-indifference hypothesis, again leaves us short of a closed form solution.

The fourth scenario takes us back to the original setting of endogenous barriers and a non-zero quadratic penalty function, coupled with the assumptions of zero drift and symmetric transaction costs ($f_a = f_b, v_a = v_b$). It can be shown that symmetry of the optimal barriers ($a^* = -b^*$) and return points ($x_a^* = -x_b^*$) about the static-optimal leverage is a corollary of the assumption of full symmetry of the penalty function, transaction costs, and underlying process about $x_0 = 0$. The objective function now becomes

$$\begin{aligned} Z_4(a, x_a, x_b, b) &= \frac{\sigma^2 C_b + \frac{\zeta}{12}(b^4 - x_b^4)}{(b - x_b)(b + x_b)} \\ &= \frac{\sigma^2(f_b + v_b(b - x_b)) + \frac{\zeta}{12}(b^4 - x_b^4)}{(b - x_b)(b + x_b)} \end{aligned} \tag{18}$$

where $a$, $x_a$, $f_a$, and $v_a$ dropped out because of symmetry. Minimization of this function with respect to $b$ and $x_b$ yields an implicit solution arrived at in steps.

The following equation is first solved for $j$:

$$4\rho^2 f_b^3 = f_b j^8 + v_b j^9 \tag{19}$$

where $\rho = 6\sigma^2/\zeta$, and then uniquely solved for $x_b^*$ (equals $-x_a^*$):

$$x_b^* = \frac{j(-2\rho f_b^2 + f_b j^4 + v_b j^5)}{4\rho f_b^2} \tag{20}$$

and for $b^*$ (equals $-a^*$) using the definition

$$b^* = j + x_b^* \tag{21}$$

In a simplified fifth scenario of no variable costs, we obtain the closed form solution

$$b^* = \left(\frac{12\sigma^2 f_b}{\zeta}\right)^{1/4}, \quad x_b^* = 0 \tag{22}$$

(equal respectively to $-a^*$ and $x_a^*$). It is worth noting that because of symmetry about $x_0 = 0$, the equality $x_b^* = 0$ implies $x_a^* = 0$, so that the upper and lower return points converge. This result, which confirms the results of BGY, is consistent with the general theory developed by Scarf (1960) and Iglehart (1963).

To obtain a closed form solution with both fixed and variable transaction costs, we further restrict the third scenario described above by the assumption of symmetric transaction costs on top of the assumptions of zero drift, binding exogenous barriers, and a flat penalty function. The objective function of this sixth scenario, which is consistent with the leverage-indifference hypothesis, is derived from (18) by substituting $\zeta = 0$ and treating the symmetric barriers as exogenous. Minimization with respect to $x_b$ yields the explicit unique solution for $x_b^*$ and, symmetrically, $x_a^*$:

$$x_b^* = \left(\frac{f_b}{v_b} + b\right) - \left(\left(\frac{f_b}{v_b} + b\right)^2 - b^2\right)^{1/2} = -x_a^* \tag{23}$$

The presence of variable transaction costs ($v_a = v_b > 0$) ensures the divergence of the two optimal return points, $x_a^* < x_b^*$ (hence $x_a^* = -x_b^*$). The presence of fixed transaction costs ($f_a = f_b > 0$) further ensures the divergences $x_a^* > a$ and $x_b^* < b$ (here $x_a^* - a = b - x_b^*$), so that $a < x_a^* < x_b^* < b$. The combinations of assumptions facilitating analytical solutions of the optimal refinancing strategy under the fifth and sixth scenarios are listed in Table 1 side by side with those of scenarios described earlier which allow only numerical solutions.

The divergence of the optimal return points in (20) and (23) is an important result which can be intuitively understood by reference to the extreme cases of pure fixed transaction costs versus pure variable transaction costs. With only fixed transaction costs at both barriers, the optimal transaction at either barrier would readjust the leverage to the same point between the barriers at which the expected penalty and transaction costs until and including the next encounter with either barrier is at a minimum. With only variable transaction costs at both barriers, the optimal policy

**Table 1.** Combination of features in the six scenarios modelled.

| *Feature* | *Scenario* General case | *1st* | *2nd* | *3rd* | *4th* | *5th* | *6th* |
|---|---|---|---|---|---|---|---|
| Drift | Yes | Yes | No | No | No | No | No |
| Penalty function | Any | Quad. symm. | Quad. symm. | Flat | Quad. symm. | Quad. symm. | Flat |
| Transaction costs | Any | Any | Any | Any | Linear symm. | Fixed symm. | Linear symm. |
| Barriers | Endo. | Exog. | Endo. | Exog. | Endo. | Exog. | Exog. |
| Return points | Two endo. | Two endo. | Two endo. | Two endo. | Two endo. | One endo. | Two endo. |
| Analytic solution | No | No | No | No | No | Yes | Yes |

at either barrier would call for infinitesimal refinancing steps so as to keep the leverage index from crossing the barrier – expecting the diffusion process to pull the leverage index away from the barrier sooner or later. This policy has two optimal return points, one at each barrier. The combination of fixed and variable transaction costs pits the forces behind these extreme policies against each other: the greater the relative weight of variable costs, the closer the two return points would be to their respective barriers, lying farther apart from each other. Note that the cost structures at the two barriers need not be similar. For example, to the extent that stock repurchase at the lower barrier takes the form of incremental open market transactions which may entail only variable transaction costs, the optimal strategy would call for infinitesimal refinancing steps at that barrier ($x_a^* = a$), but a lump-sum issuance transaction at the upper barrier ($x_b^* < b$).

The firm's excess cost of capital is obtained from the minimum cost associated with the unique optimal refinancing strategy. In the sixth scenario, that cost is specified by substituting in (18) the value of $\zeta = 0$, the exogenous parameter $b$, and the optimal $x_b^*$ based on (23):

$$Z_6^*(x_a, x_b) = \frac{\frac{1}{2}\sigma^2 v_b^2}{f_b + bv_b - (f_b^2 + 2bf_b v_b)^{\frac{1}{2}}} \tag{24}$$

Having been derived under the assumption of a zero penalty, this excess capital cost is attributed entirely to linear transaction costs occurring stochastically in a dynamic setting. The minimum excess periodic cost, $Z_6^*$, must be divided by the value of the firm to express the excess cost of capital as a rate. In the fifth scenario, the excess cost of capital can be derived in a similar manner by substituting into (18) the optimal numerical values of $x_b^*$ and $b^*$ obtained from (20) and (21).

## 4. A positive refinancing theory

An analytical solution for the optimal refinancing strategy provides at once a basis for a positive dynamic leverage theory and the means for testing it. The following relationships implied by our optimal solution can be tested as partial equilibrium relationships consistent with the sixth scenario.

## 4.1 *Refinancing frequency*

The optimal expected time interval between transactions at either barrier is calculated based on (7) and (13). In the sixth scenario of dual return points, the assumption of full symmetry including zero drift causes (6) to take the simplified form

$$U(x) = \frac{b+x}{2b} \tag{6'}$$

so that (7) is reduced to

$$V(x) = \frac{b^2 - x^2}{\sigma^2} \tag{7'}$$

while (13a) and (13b) assume the following value

$$P^*(a) = P^*(b) = \frac{1}{2} \tag{13'}$$

Based on these results, the optimal expected interval between transactions is

$$\begin{aligned}\overline{V}^* &= P^*(a)V^*(x_a) + P^*(b)V^*(x_b)\\ &= \frac{b^2 - x_b^{*2}}{\sigma^2}\\ &= \frac{b^2 - \left\{b + \frac{f_b}{v_b} - \left[\left(b + \frac{f_b}{v_b}\right)^2 - b^2\right]^{1/2}\right\}^2}{\sigma^2}\end{aligned} \tag{25}$$

Under the full symmetry and single return point of the fifth scenario, (25) is reduced to

$$\overline{V}^* = \frac{b^2}{\sigma^2} \tag{26}$$

confirming the results of BGY for the special case of purely fixed transaction costs.

## 4.2 *The mean leverage*

The mean leverage concept has so far remained outside the analysis. It is apparent from the above derivations that the mean is not a decision parameter of the firm's leverage strategy, but a by-product of that strategy. The mean plays a prominent role in past empirical research, often as a proxy for the unobserved static-optimal leverage. The mean leverage implied by the optimal strategy, $\bar{x}^*$, is derived by first re-evaluating the integrals in (9) after substituting $x$ for $g(x)$ and specifying the optimal $a^*$, $b^*$, and $x_a^*$ or $x_b^*$. The resulting temporary values $y^*(x_a)$ and $y^*(x_b)$ are then divided, respectively, by the optimal time intervals $V^*(x_a)$ and $V^*(x_b)$ to generate $Y^*(x_a) = y^*(x_a)/V^*(x_a)$ and $Y^*(x_b) = y^*(x_b)/V^*(x_b)$, which are then used together with the probabilities stated in (13) to calculate

$$\bar{x}^* = P^*(a)Y^*(x_a) + P^*(b)Y^*(x_b) \tag{27}$$

With full symmetry about the static-optimal leverage, $x_0 = 0$, the sixth scenario yields the expected solution $\bar{x}^* = 0$. A similar solution would be obtained in the fifth scenario if $f_b = f_a$ and $v_b = v_a$ were numerically specified. Consistent with Fig. 1, it is noteworthy that the mean leverage is generally unequal to the static-optimal leverage or the mid-point between the barriers or return points, and may not lie between the return points.

## 4.3 *Elasticities*

Testable theoretical relationships in the sixth scenario are provided by the following partial elasticities of the optimal return points, excess capital cost, and expected time interval between refinancing transactions with respect to the fixed and variable unit transaction costs and the variance of the state variable.

Based on (23), we derive

$$\eta(x_b^*, f_b) = -\left(1 + \frac{2bv_b}{f_b}\right)^{-1/2} \tag{28}$$

and with the opposite sign

$$\eta(x_b^*, v_b) = -\eta(x_b^*, f_b) \tag{29}$$

Given $b$, the variance does not affect the optimal return point, so that

$$\eta(x_b^*, \sigma^2) = 0 \tag{30}$$

Based on a simplified version of (18) in which $\zeta = 0$ and the symmetric $b = -a$ are exogenous, we obtain from (24)

$$\eta(Z_6^*, f_b) = \left(1 + \frac{2bv_b}{f_b}\right)^{-1/2} \tag{31}$$

$$\eta(Z_6^*, v_b) = 1 - \left(1 + \frac{2bv_b}{f_b}\right)^{-1/2} \tag{32}$$

both of which are positive, as expected, and less than unity.

Finally, based on (26), we derive

$$\eta(\overline{V}^*, f_b) = \frac{2f_b^3 + 4bf_b^2v_b + b^2f_bv_b^2 - 2(f_b^2 + bf_bv_b)[f_b(f_b + 2bv_b)]^{1/2}}{f_b^3 + 3bf_b^2v_b + 2b^2f_bv_b^2 - [f_b(f_b + 2bv_b)]^{3/2}} \tag{33}$$

and by reversing the sign

$$\eta(\overline{V}^*, v_b) = -\eta(\overline{V}^*, f_b) \tag{34}$$

The partial elasticity with respect to the variance can be derived from (26):

$$\eta(\overline{V}^*, \sigma^2) = -1 \tag{35}$$

which has the expected negative sign.

### 4.4 *Broad empirical implications*

The general case of refinancing with fixed and variable transaction costs and dual return points has empirical implications that go beyond the specific results derived for the sixth scenario.

First, any unique static-optimal leverage would be generally distinct from the optimal dual return points and the leverage mean, and could not be empirically estimated from those values. Numerous studies attempt such estimation based on the assumption that dynamic deviations from the static-optimal leverage can be treated as random errors with a zero mean.

Second, a firm's policy tolerating wide fluctuations of the financial leverage cannot be interpreted as evidence of a flat-bottom static cost of capital function. Such a policy is generally consistent with the existence of a unique static-optimal leverage in the presence of substantial refinancing costs.

Third, to correctly estimate the overall leverage mean, it is necessary to take into account the effect of the starting (return) point on the potential time paths, and therefore on the conditional mean. The overall mean is the expectation of the two conditional means.

Fourth, leverage time series are likely to display positive serial correlation because a start at the upper (lower) return point is more likely to be followed by a return to the same point. Stock issuance is more likely to be followed by issuance than by repurchase, and vice versa.

Fifth, under the optimal strategy, no leverage adjustment is made until a barrier is reached, and – due to fixed transaction costs – full adjustment from the barrier to the relevant return point is a single event executed in lump sum. This aspect of the optimal strategy is confirmed by empirical studies which generally fail to uncover a systematic reversion of the leverage to its mean or any other point.

Sixth, static estimates of the firm's cost of capital contain biases from two sources: (a) refinancing transaction costs are probabilistic rather than deterministic, and should be measured under the optimal refinancing strategy; and (b) the penalty associated with tolerable random excursions from the static-optimal leverage should be included in the cost of capital.

## 5. Numerical solution

The class of dynamic leverage models presented above may be further generalized when the search for closed form solutions is abandoned, and numerical methods are used to generate optimal strategies under realistic conditions with the desired degree of complexity. This flexibility may be especially useful for empiricists and practitioners. In the eight numerical solutions presented in Table 2, the leverage is measured by the debt/asset ratio of a firm with one dollar of assets. The base case utilizes a bifurcated, asymmetric U-shaped penalty function, $g(x)$, created from two power functions:

$$g(x) = \frac{\zeta_{left}}{2}(x_0 - x)^{\alpha_{left}} + \eta \quad \text{when } x < x_0$$
$$= \frac{\zeta_{right}}{2}(x - x_0)^{\alpha_{right}} + \eta \quad \text{otherwise,} \tag{36}$$

where the power parameters are $\alpha_{left} = 1.7$ and $\alpha_{right} = 2$, and the slope coefficients are $\zeta_{left} = 0.426$ and $\zeta_{right} = 3$. These parameters are selected to produce a realistic penalty function with a unique minimum cost of capital of $\eta = 15\%$, occurring when the debt ratio is $x_0 = 50\%$, and with static costs of capital of 15.5% and 16.5% occurring at debt ratios of $x = 40\%$ and $x = 60\%$, respectively. The instantaneous annual drift and volatility parameters of the process are $\mu = 0.1$ and $\sigma = 0.2$, the fixed costs per equity transaction are $f_a = \$0.005$ and $f_b = \$0.05$, and the variable cost coefficients are $v_a = 0.01$ and $v_b = 0.025$.

The solution for the base case, reported under the same column, shows a wide optimal leverage range between $a^* = 9.80\%$ and $b^* = 73.87\%$ with respective return points $x_a^* = 29.59\%$ and $x_b^* = 32.60\%$ which are distinct but not far apart. The distance $x_0 - a^* = 40.2\%$ is greater than the distance $b^* - x_0 = 23.87\%$ because of the higher fixed transaction cost at the upper barrier and the positive drift, effects which are mitigated by the greater penalty assumed for excess debt than for debt deficiency. The optimal size of stock issues is greater than that of stock repurchases, as

**Table 2.** Numerical solution for cases with exogenous parameters $x_0 = 50\%$ and $\eta = 15\%$ for a firm with one dollar assets.

| | *Base case* | *Case A* | *Case B* | *Case C* | *Case D* | *Case E* | *Case F* | *Case G* |
|---|---|---|---|---|---|---|---|---|
| Input | | | | | | | | |
| Standard deviation, $\sigma$ | 0.200 | 0.100 | | | | | | |
| Drift, $\mu$ | 0.100 | | 0.050 | | | | | |
| Penalty exponent, $\alpha_{left}$ | 1.700 | | | | | | | |
| Penalty slope factor, $\zeta_{left}$ | 0.426 | | | 1.278 | | | | |
| Penalty exponent, $\alpha_{right}$ | 2.000 | | | | | | | |
| Penalty slope factor, $\zeta_{right}$ | 3.000 | | | | | | | |
| Fixed repurchase cost \$, $f_a$ | 0.005 | | | | 0.010 | 0.0075 | | |
| Variable repurchase cost, $v_a$ | 0.010 | | | | | | 0.015 | 0.000 |
| Fixed issuance cost \$, $f_b$ | 0.050 | | | | | 0.075 | | |
| Variable issuance cost, $v_b$ | 0.025 | | | | | | 0.0375 | 0.000 |
| Output | | | | | | | | |
| Lower debt barrier, $a^*$ % | 9.80 | 14.91 | 12.16 | 23.33 | 5.51 | 3.77 | 8.63 | 11.11 |
| Lower return point, $x_a^*$% | 29.59 | 27.24 | 31.18 | 37.84 | 28.81 | 24.47 | 28.85 | 30.04 |
| Upper return point, $x_b^*$% | 32.60 | 29.90 | 35.69 | 40.65 | 32.83 | 28.43 | 33.14 | 30.04 |
| Upper debt barrier, $b^*$% | 73.87 | 66.41 | 74.74 | 76.16 | 74.01 | 76.12 | 73.99 | 73.61 |
| Mean leverage, $\bar{x}^*$% | 42.31 | 43.66 | 42.14 | 49.04 | 41.54 | 40.53 | 42.20 | 41.93 |
| Capital cost, $z^*$% | 18.29 | 17.56 | 17.82 | 19.31 | 18.34 | 19.00 | 18.45 | 17.93 |
| Refinancing interval, $\bar{v}^*$ yr | 2.25 | 3.40 | 2.26 | 1.53 | 2.54 | 2.78 | 2.34 | 2.11 |

reflected in the effect on the firm's debt ratio: $b^* - x_b^* > x_a^* - a^*$. The transaction size is directly related to the fixed transaction cost and inversely to the variable cost parameter. The close distance between the return points themselves, $x_b^* - x_a^* = 3.01\%$, is caused by the relatively small component of variable costs.

Due to lack of overall symmetry in the process and the various costs, the mean leverage associated with the optimal strategy, $\bar{x}^* = 42.31\%$, is not at the half-way point between the barriers (41.84%) or the return points (31.10%).

The dynamic-minimal cost of capital, $z^* = 18.29\%$, is 3.29% greater than the cost of capital of 15% attributed to the optimal leverage in the static context. The optimal mean time between equity transactions is 2.25 years.

The partial effects of the various parameters on the optimal strategy is brought to focus by comparing the base case with cases A through G in which individual parameters are changed one or two at a time.

*Case A*. The primary effect of a decrease of the variance from $\sigma = 0.2$ to $\sigma = 0.1$ is a decrease of the optimal distance between the barriers.

*Case B*. A decrease of the drift from $\mu = 0.1$ to $\mu = 0.05$ pushes both barriers upward.

*Case C*. An increase in the slope of the left branch of the penalty function from $\zeta_{left} = 0.426$ to $\zeta_{left} = 1.278$ (so that the penalty at $x = 0.4$ increases from $g(x) = 15.5\%$ to $g(x) = 16.5\%$) causes a dramatic upward shift of the lower barrier from $a^* = 9.80\%$ to $a^* = 23.33\%$ accompanied by substantial shifts in the same direction of $x_a^*$, $x_b^*$, $b^*$, and $\bar{x}^*$. These changes cause a measurable increase of the cost of capital from $Z^* = 18.29\%$ to $Z^* = 19.31\%$ and a decrease of the expected time between equity transactions from $\overline{V}^* = 2.25$ years to $\overline{V}^* = 1.53$ years.

*Case D*. An increase of the fixed transaction cost component of stock repurchase from $f_a = \$0.005$ to \$0.010 mainly causes a decrease of the optimal lower barrier from 9.80% to 5.55%.

*Case E*. An equal proportional increase of 50% in fixed equity transaction costs for stock repurchase and issuance pushes both barriers out and increases the cost of capital from $z^* = 18.29\%$ to 19.00%. A relative increase of fixed transaction costs tends to increase the optimal size of equity transactions.

*Case F*. An equal proportional increase of 50% in variable equity transaction costs for stock repurchase and issuance causes a small upward shift of the return points. A relative increase in variable transaction costs tends to decrease the optimal size of equity transactions.

*Case G*. As an extension of Case F, the elimination of variable equity transaction costs causes the upper and lower return points to converge at $x_a^* = x_b^* = 30.04\%$. The absence of variable transaction costs is a special case of the class of models presented in this paper.

## 6. Conclusion

This study was motivated by the mounting evidence that the financing of firms in the real world is dominated by a hierarchy based on differential transaction costs, a behaviour which is not explained by traditional theories of static tradeoff. We offer a class of diffusion models that mimic this behaviour in a stochastic dynamic framework and are designed to optimize a financing strategy using a wide range of static tradeoff theories as input. Our models accommodate a combination of fixed and variable refinancing costs which generally lead to dual target debt ratios – one for each of the refinancing barriers. We obtain new analytical expressions consistent with an optimal refinancing strategy, including the firm's cost of capital and its mean leverage. We also demonstrate the flexibility of our class of models in handling complex scenarios which are not amenable to an analytical solution. Our class of models augments the pecking order theory and provides a quantitative framework for its implementation as a decision tool. It also provides additional hypotheses for its empirical validation. Symmetrically, our results show the importance of dynamic factors in designing and interpreting empirical tests of static tradeoff theories.

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# Appendix A

## Steady-state probabilities of starting the process at $x_a$ versus $x_b$

Since any encounter with barrier $a(b)$ triggers refinancing and shifting of the state variable to return point $x_a(x_b)$, the probability of starting the process at $x_a(x_b)$ is the same as the probability of hitting first barrier $a(b)$, denoted by $P(a)[P(b)]$. Let $\mathrm{d}(x_a)[\mathrm{d}(x_b)]$ denote the conditional probability of hitting first the lower barrier, $a$, if the process starts at $x_a(x_b)$, and $u(x_a)[u(x_b)]$ denote the conditional probability of hitting first the upper barrier, $b$, if the process starts at $x_a(x_b)$. Since the following relationships must hold in any refinancing cycle:

$$\mathrm{d}(x_a) + u(x_a) = 1 \tag{A1}$$

$$\mathrm{d}(x_b) + u(x_b) = 1 \tag{A2}$$

$$P(a) + P(b) = 1 \tag{A3}$$

we can express sequentially the probabilities of hitting first each barrier in cycle $t+1$ as

$$P(a)_{t+1} = P(a)_t \mathrm{d}(x_a)_t + P(b)_t \mathrm{d}(x_b)_t \tag{A4}$$

$$P(b)_{t+1} = P(b)_t u(x_b)_t + P(a)_t u(x_a)_t \tag{A5}$$

Recalling the assumption of steady state, we drop the subscripts and rewrite

$$P(a) = P(a)\mathrm{d}(x_a) + (1 - P(a))\mathrm{d}(x_b) \tag{A6}$$

$$P(b) = P(b)u(x_b) + (1 - P(b))u(x_a) \tag{A7}$$

When these equations are solved respectively for $P(a)$ and $P(b)$, we obtain the expression appearing in Equations (13a) and (13b) in the text.